A unique signum switch for chaos and hyperchaos

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Abstract:

A new piecewise-linear element with the form of *x*sign(*y*) is applied for chaos and hyperchaos. By the use of some diodes, resistors and operational amplifiers, a unique signum element with a signal line and a control line is designed to take the place of a multiplier and realize switchable piecewise-linearity. Correspondingly, the hyperchaotic circuit is designed without multipliers or inductors, which are present in other hyperchaotic circuits, and thus this circuit has not been previously reported.

Key words: Piecewise-linear; signal line; control line; hyperchaos.

1. Introduction

It is well known that nonlinearity is necessary for a system to display chaos or hyperchaos. Sprott have proposed a series of well-known simple chaotic systems [1-5], which correspondingly lead to simple chaotic circuits. The simplest chaotic circuits are constructed from the systems with a signum or an absolute-value element [6]. In general, it is difficult to achieve hyperchaos only using signum or absolute-value nonlinearities, and that is why many of the hyperchaotic systems were constructed using extensions of the Lorenz and Rössler systems, in which multipliers are required to realize the quadratic nonlinearities [7-13]. Other piecewise-linear hyperchaotic systems [14-15] look simple, but these systems rely on new circuit elements of negative resistors and inductors, which make the circuit complicated and unsuitable for integration. Chlouverakis and Sprott [16] proposed what may be the algebraically simplest hyperchaotic snap system, but it contains a fifth-order nonlinearity that requires four multipliers to realize.

However, there is a useful type of piecewise-linear factor besides the absolute-value and signum functions, which has the form of *x*sgn(*y*), and the nonlinearity comes from the polarity of *y*. Even though this kind of nonlinearity can be realized with a multiplier with the input signals *x* and sgn(*y*), it can also be achieved with a switch, where the polarity of the variable *y* determines whether the output is *x* or –*x*. If *y* is positive, the signal *x* is selected, and if *y* is negative, the signal –*x* is selected. The special piecewise-linear element has value in nonlinear dynamics since cross product terms play a key role in attractor forming and may lead a system to be chaotic or hyperchaotic.

When only the polarity information of any of the variables in a cross-product term is retained, the quadratic term is a special piecewise-linear term with a signum nonlinearity. In this paper, a new piecewise-linear hyperchaotic system is constructed based on this polarity information, which has a simple nonlinearity without any quadratic or higher-order polynomials. In Section 2, we describe the piecewise-linear element using the signum function, and by this method, the wide-known diffusionless Lorenz system is transformed into a new piecewise-linear system. By further dimension extension, a new 4-D hyperchaotic system with seven terms is constructed. In Section 3, the corresponding chaotic and hyperchaotic circuits are designed with diodes for the signal switch. Some discussion and conclusions are contained in the last section.

2. Chaos and hyperchaos with piecewise-linearity using polarity information

When the Lorenz system is rescaled as $(x, y, z) \to (\sigma x, \sigma y, \sigma z + r)$, $t \to t/\sigma$ and taking $r, \sigma \to \infty$ while R = br/σ^2 remains finite, the diffusionless Lorenz system [17-18] results,

$$\begin{cases} \dot{x} = y - x, \\ \dot{y} = -xz, \\ \dot{z} = xy - R, \end{cases} \tag{1}$$

It happens that the chaos is preserved if the quadratic terms *xz* and *xy* in this equation are replaced with signum nonlinearities -*z*sgn(*x*) and *x*sgn(*y*), resulting in the system

$$\begin{cases} \dot{x} = y - x, \\ \dot{y} = -z\,\mathrm{sgn}(x), \\ \dot{z} = x\,\mathrm{sgn}(y) - a, \end{cases} \tag{2}$$

Here polarity information is injected into the quadratic terms in the second and third dimension, where the quadratic nonlinearity turns into a piecewise-linearity, and thus makes a piecewise-linear system, where the only constant parameter *a* ceases to be a bifurcation control and becomes an amplitude parameter [19-20]. Correspondingly, the parameter *a* can be set to unity without loss of generality, in which case system (2) gives a chaotic attractor with Lyapunov exponents of (0.131, 0, –1.131), which resembles the original diffusionless Lorenz system but with discontinuities in the direction of the flow vector as shown in Fig. 1.

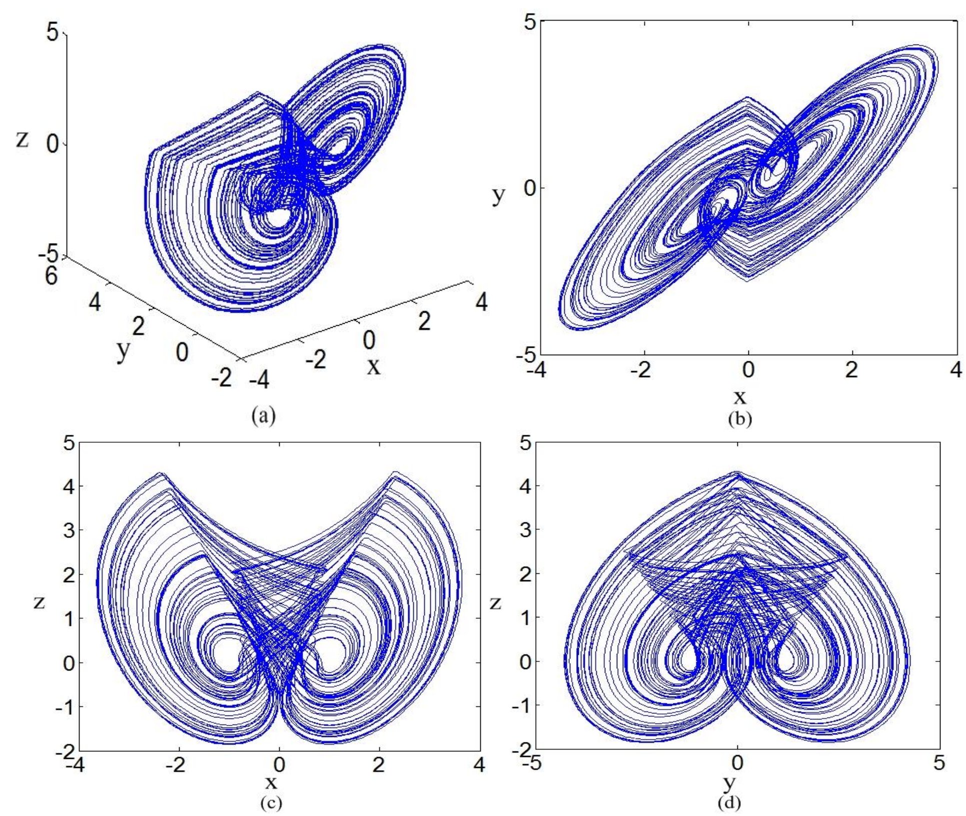


Fig. 1 Chaotic attractor from Eq. (2) for initial conditions (0, 1, 0) with LEs (0.131, 0, -1.131), (a) three-dimensional view, (b) *x*-*y* phase plane, (c) *x*-*z* phase plane, (d) *y*-*z* phase plane.

By introducing linear feedback from an additional dimension (u) in system (2), a 4-D piecewise-linear system with signum nonlinearities is obtained as follows,

$$\begin{cases} \dot{x} = y - x, \\ \dot{y} = -z\,\mathrm{sgn}(x) + u, \\ \dot{z} = x\,\mathrm{sgn}(y) - a, \\ \dot{u} = -by, \end{cases} \tag{3}$$

Like system (2), the above system 4-D system also has amplitude parameter a and a new parameter b that gives bifurcations as shown in Fig. 2.

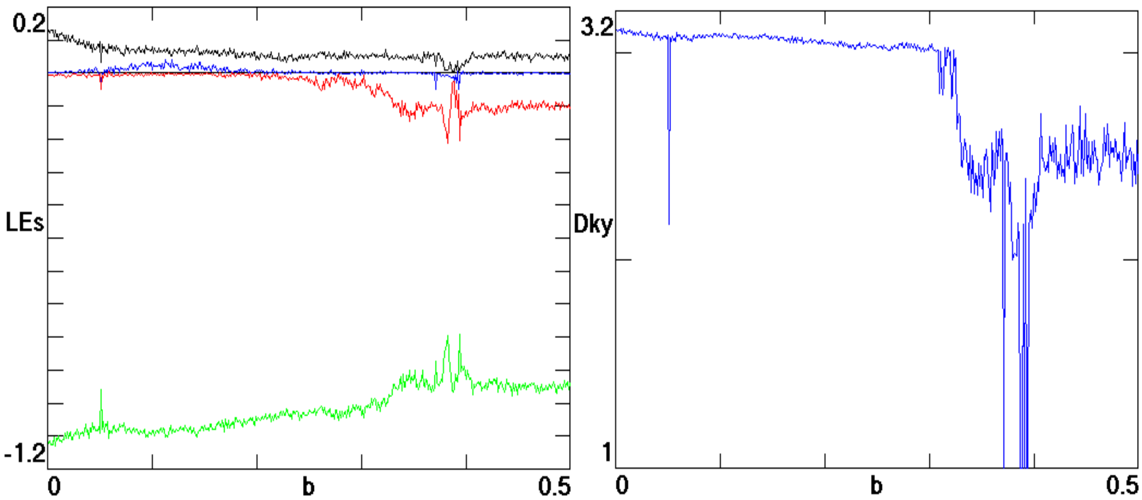
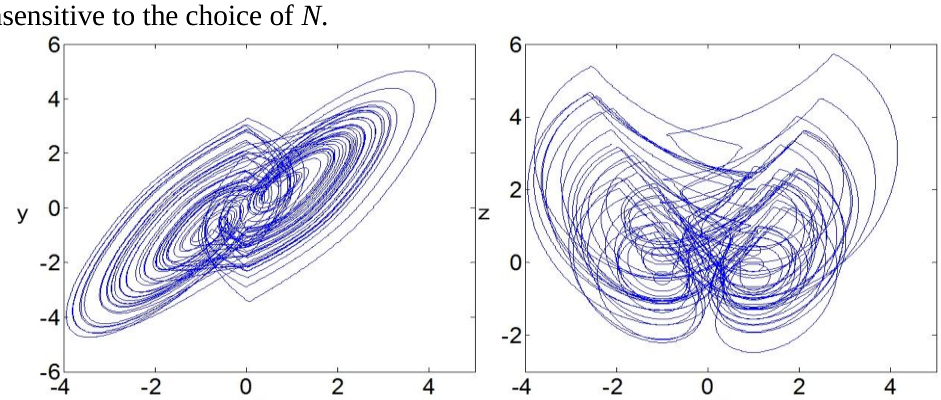
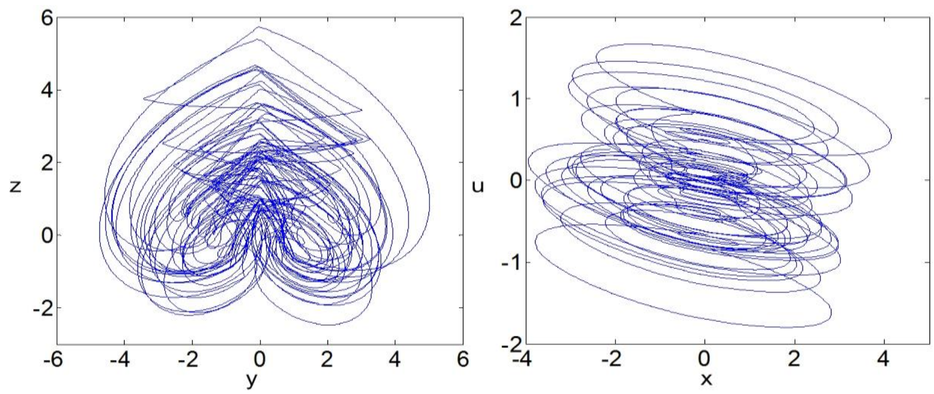


Fig.2. Lyapunov exponents of system (3) versus b at a = 1 and the corresponding Kaplan-Yorke dimension.

In the region of b in (0, 0.5], system (3) is predominantly chaotic with some hyperchaotic and periodic windows. For example, when b = 0.38, the system displays a symmetric limit cycle. When a =1, b = 0.1, system (3) is hyperchaotic with Lyapunov exponents of (0.071, 0.022, 0, –1.089) with a hyperchaotic attractor as shown in Fig. 3. The corresponding Poincaré section in the hyperchaotic region has a dimension at least 2.0 as shown in Fig. 4. Since the signum function is not continuous, calculation of the Lyapunov exponents is problematic, but sgn(x) can be replaced by a smooth approximation given by tanh(Nx) [21-22] with N large without any significant change in the attractor. We believe the calculated Lyapunov exponents are reliable for N = 250 if the maximum Runge-Kutta step size is 0.0005 since the values are insensitive to the choice of N.



(a) (b)



(c) (d)

Fig. 3 Hyperchaotic attractor observed from system (3) for initial conditions (1, 0, 3, 0) (a) *x*-*y* phase plane, (b) *x*-*z* phase plane, (c) *y*-*z* phase plane, (d) *x*-*u* phase plane.

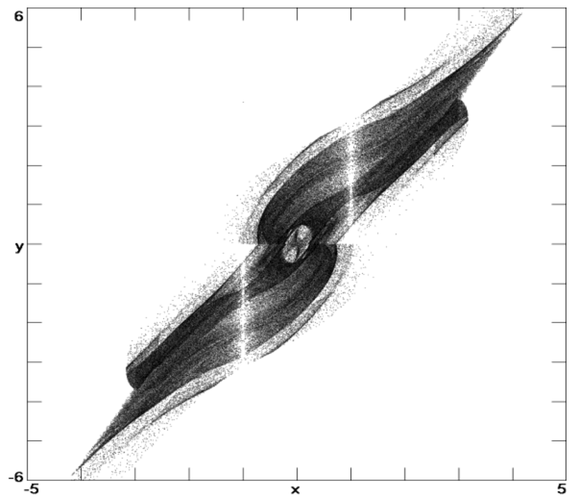


Fig. 4 Projection onto the *x*-*y* plane of a cross-section of the attractor at $z = 0$ for system (3).

The equations of systems (2) and (3) are simple and elegant. The parameter *a* is an amplitude parameter, which determines the size of the attractor. When *a* is positive, system (2) has one equilibrium point at (a, a, 0), which is different from the diffusionless Lorenz system with two equilibrium points. System (3) has no equilibrium points, and thus the resulting periodic, chaotic, and hyperchaotic attractors are "hidden" in the sense that they cannot be found by using an initial condition in the vicinity of an unstable equilibrium [23-24]. However, unlike the system with quadratic nonlinearities [10], the attractor of system (3) is globally attracting (all initial conditions approach it), and so as a practical matter it could hardly be less hidden. These are important features for the operation of the circuit, since the initial capacitor voltages do not matter, it cannot fail to oscillate, and there is no danger of it saturating by unbounded growth of the signals as is typical of other chaotic circuits with finite and often small basins of attraction.

Since the rate of volume contraction is –1, systems (2) and (3) are dissipative with solutions as time goes to infinity that contract onto an attractor of zero measure in their state space. Both systems have rotational symmetry with respect to the *z*-axis as evidenced by their invariance under the coordinate transformation $(x, y, z) \rightarrow (-x, -y, z)$ or $(x, y, z, u) \rightarrow (-x, -y, z, -u)$.

3. Novel structure of switch element with a signal line and a control line

In this section, a new switch element is designed and constructed to realize the polarity reversal associated with the signum function. For the signum nonlinearities z sgn(x) and $-x$ sgn(y), the most common electronic implementation method uses multipliers [25-26] such as the example in Fig. 5. An alternative is to use a circuit that can switch when the polarity of a signal changes. The circuit for the implementation of z sgn(x) will select signal z or $-z$ depending on the polarity of x and send it to the integration channel. Such a circuit can be built with just diodes, resistors, and a few extra operational amplifiers.

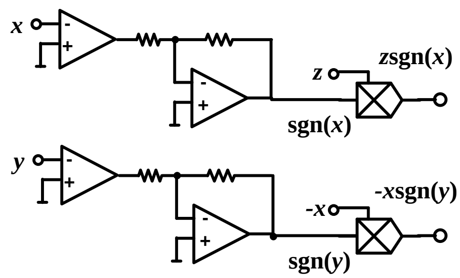


Fig. 5 Multipliers to realize the signum operation

The schematic for this switching circuit is shown in Fig. 6. Signals z and $-z$ pass through a signal line and can be blocked when a voltage source is applied from a control line. Whether the control line will apply this voltage depends on the polarity of the signal x. If x is positive, signal z is blocked and $-z$ will pass through. If it is negative, then $-z$ is blocked and z passes through. If it is zero, both signals z and $-z$ sum to zero. Two independent signals z and $-z$ are selected by the control line to pass, whereas each signal (z or $-z$) occupies two separate signal lines according to its polarity. Another switch with a signal line and a control line realizes the function x sgn(y). All signal lines have their own independent voltage followers to provide impedance matching.

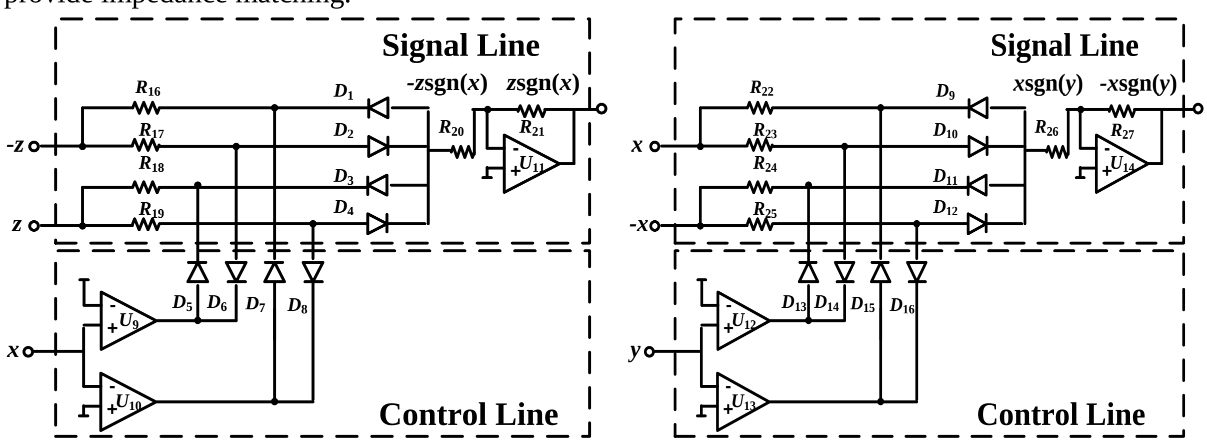


Fig. 6 signal line and control line combined to realize the signum operation

Four integration channels and the above circuit that models the signum nonlinearity to realize system (3), as shown in Fig. 7. The integration channels are designed by general analog computation methods. The complete circuit of Figs. 6 and 7 will produce a hyperchaotic signal. When U_7, U_8, C_4, R_5, R_{13}, R_{14} and R_{15} are removed, it realizes system (2) and produces a chaotic signal. Oscilloscope traces from the output of the integration channels are given in Figs. 8 and 9. The circuit parameters are $C_1=C_2=C_3=C_4=1$nF, $R_1=R_2=R_3=R_4=R_5=R_7=R_8=R_{11}=R_{12}=$ $R_{14}=R_{15}$ $=100\text{k}\Omega$, $R_{20}=R_{21}=R_{26}=R_{27}=200\text{k}\Omega$,

R_{16}=R_{17}=R_{18}=R_{19}=R_{22}=R_{23}=R_{24}=R_{25}=10kΩ, R_6=R_{10}=100kΩ, R_9=900kΩ, R_{13}=1MΩ. The operational amplifiers are TL084 ICs powered by ±9 volts. Germanium diodes 1n60p were used to reduce the influence of threshold voltage and crossover distortion. Other diodes such as the 1n4148 will not give an attractor that is in such close agreement with numerical results. Operational amplifiers U_{11} and U_{14} are required to provide a current sufficient to drive the corresponding resistors R_6 and R_{10}.

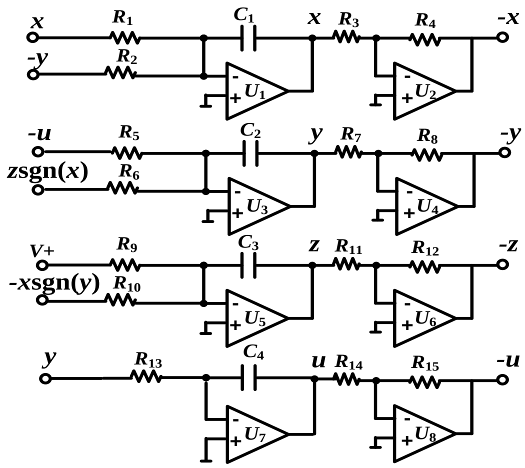
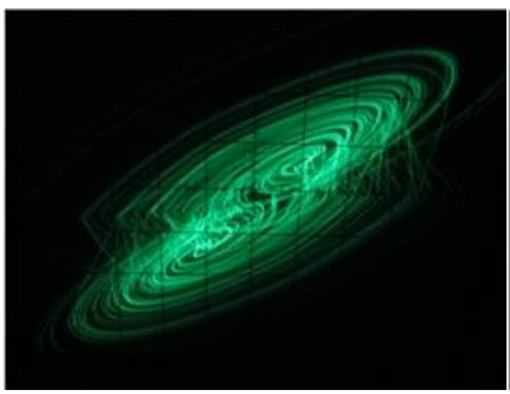
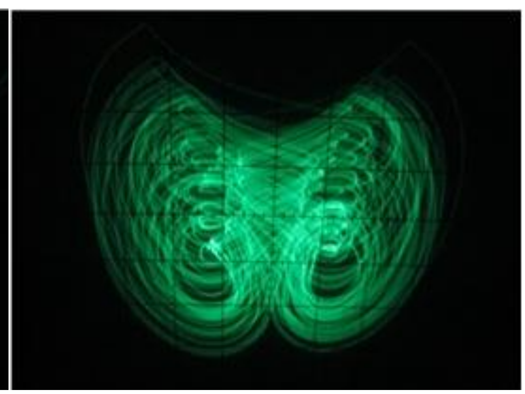


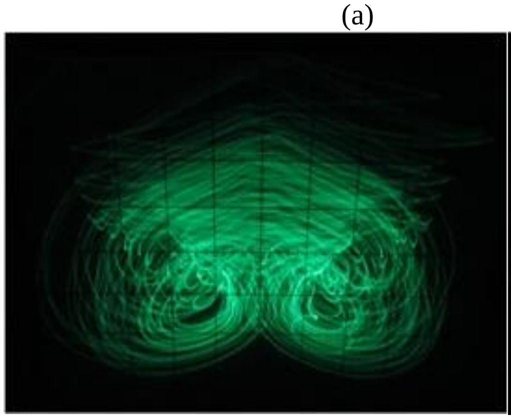
Fig. 7 Four integration channels in circuit structure for the 4-D system (3).



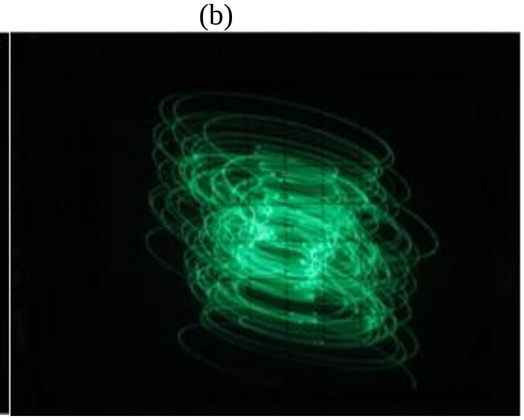
(a)



(b)



(c)



(d)

Fig.8 Oscilloscope traces of hyperchaotic attractors of system (3) (a) *x*-*y* plane, (b) *x*-*z* plane, (c) *y*-*z* plane, (d) *x*-*u* plane (1V/div).

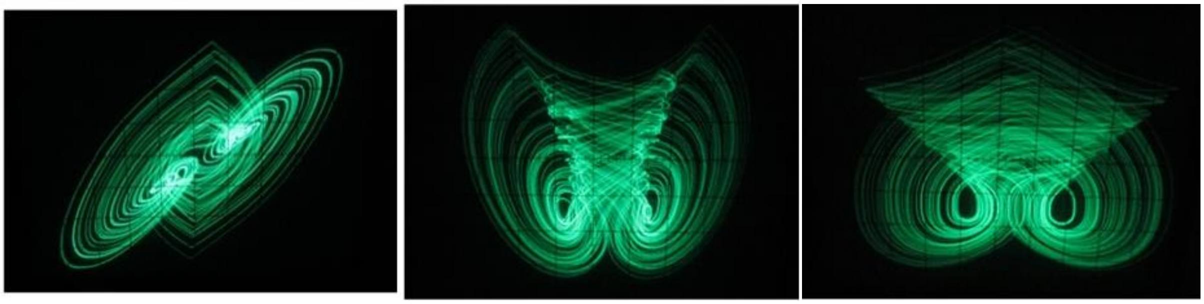


Fig.9 Experimental phase portraits of system (2) (a) *x*-*y* plane, (b) *x*-*z* plane, (c) *y*-*z* plane (1V/div).

The number of components in the switching circuit can be further reduced by changing the piecewise nonlinearities to remove a control line. For example, if the nonlinearities in system (3) change to $-z$ sgn(x-u) and x sgn(x-u), respectively, then the second and third integration channel can share the same control channel. This provides an alternative that will simplify the circuit, but at the expense of making the equations more complicated. However, this kind of channel multiplexing will provide chaos rather than hyperchaos under the same parameters.

4. Discussion and conclusions

The signum function can extract the polarity of variables in quadratic terms, which produces chaos and hyperchaos with a piecewise-linear nonlinearity. This approach gives rise to a new type of hyperchaotic circuit without any multipliers. To implement the piecewise-linear selection, a new structure based on a signal line and control line was designed to replace multipliers and is thus simpler than other hyperchaotic circuits with polynomial nonlinearities. The new structure of the analog signal line and control line proposed in this paper provides a valuable circuit element for realizing signum nonlinearities. The corresponding strange attractors show good agreement with numerical simulation.

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