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A simple memristive jerk system

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Abstract
A simple memristive chaotic jerk system with one variable to represent the internal state is found. The proposed equilibria-free memristive system yields hidden chaotic oscillation in a narrow parameter space. A circuit is constructed that models the jerk system, and it shows agreement with the predicted oscillation. The new memristive jerk system appears to be one of the algebraically simplest memristive chaotic systems.

1 | INTRODUCTION

Memristors have attracted great interest in engineering for their potential applications [1–5]. They have been used as a dynamic element to generate chaotic signals. The presence of a memristor usually leads to a 4-D system [6–10], although only a 3-D system is required for chaos. Several 3-D memristive chaotic systems have been reported [11–15], but in those cases, the memristor was implemented with a complicated operational-amplifier-based equivalent device.

Jerk systems are a simple type of dynamical system that can generate chaos [16–20]. Such a compact structure is composed of a couple integral operation units in series. Some jerk systems are chaotic when they contain a nonlinearity from quadratic terms [16–18], an exponential function [19] or a cubic term [20]. All other jerk systems that produce chaos based on the memristor are 4-D [21,22]. The novelty of this work is that compared with prior studies, we aim to construct a completely 3-D jerk memristive circuit. The most important challenge in this work is introducing a suitable memristor to break the existing oscillation in a 2-D structure and finally bring chaos.

In addition, many memristive systems exhibit chaotic oscillation combined with other states because of the integral effect from the memristor [6–10,21–25]. Unlike these memristive jerk systems, however, a second-order jerk structure with a memristor is explored that is both simple and robust for giving chaos. In Section 2, the model is given and analyzed with basic dynamical analysis. In Section 3, the circuit is built for proving the theoretical analysis. Finally, a short conclusion is made to summarize the work.

2 | SYSTEM MODEL

A simple chaotic jerk oscillator containing a memristor as one of the state variables was found by an exhaustive computer search based on the Euler method. Suppose there is a 2-D jerk structure $\dot{y} = z$, $\dot{z} = f(y, z)$, and introduce a flux-controlled memductance $W(x)$ in it. The following system is found for producing chaos:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -z - az^2 - W(x)y + b. \end{cases} \quad (1)$$

where the flux-controlled memductance $W(x) = 1.3x^2$-1 is introduced in the z-dot equation. Here the variables y and z are the external system variables, and x is the internal variable in

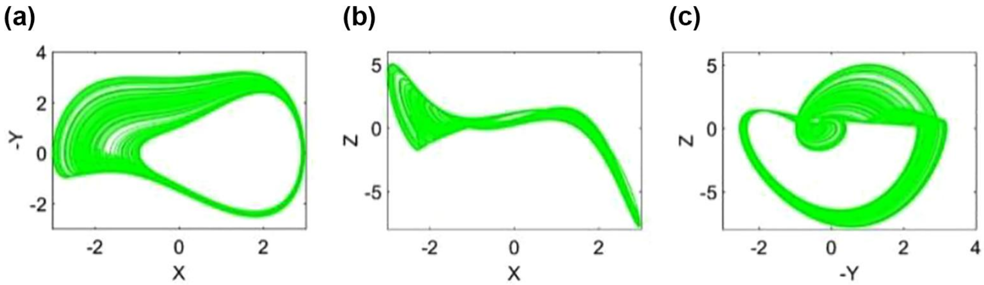

FIGURE 1 Strange attractor projections of System (1) with a = 0.239, b = 1 and initial condition (x_0, y_0, z_0) = (0, −2, −2) (a) (X, -Y) plane, (b) (X, Z) plane, (c) (-Y, Z) plane

the memristor and indicates the magnetic flux. When $a = 0.239$ and $b = 1$, System (1) produces chaos with Lyapunov exponents (0.0529, 0, −1.0529) after a time of $t = 2e7$ and a corresponding Kaplan–Yorke dimension of $D_{KY} = 2.0502$ for initial condition (0, −2, −2), as shown in Figure 1, whose basins of attraction (in the $z = 0$ plane) are shown in Figure 2.

This system is fairly delicate with chaos in only a narrow range of the parameter space, as indicated in Figure 3. The internal variable x comes from the integration of the system state variable y. System (1) is asymmetric with a speed of volume contraction determined by a derivative proposed by Lie:

$$\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -1 - 2az.$$

In addition, System (1) has no equilibria [26–30], and therefore the attractor is hidden [31–35].

More interesting is that unlike other memristive systems [8–10], System (1) outputs relatively stable oscillations except when switching between chaos and sliding initial values that agree with the plots of the dynamical region and basins of attraction. To further verify this, the solution based on offset boosting under a fixed initial condition is used for diagnosing multi-stability [36,37]. Taking offset-boosting d in the dimension $y \to y + d$, it is shown that when offset d varies in [-5, 5] except for the long transient process, System (1) remains chaotic unless it is dragged sliding with the initial condition, as shown in Figure 4.

The embedded memristor is defined as

$$\begin{cases} \dot{x} = y, \\ W(x) = 1.3x^2 \text{ - } 1, \\ i_M = W(x)y. \end{cases} \tag{2}$$

Flux-dependent memductance is related to the internal variable x, which is of quadratic degree [13,38,39] and is determined by the system variable y:

$$\begin{aligned} W(x) &= 1.3x^2 - 1 = 1.3\left(\int_{-\infty}^{t} y ds\right)^2 - 1 \\ &= 1.3\left(\int_{0}^{t} y ds\right)^2 - 1 + W_0 \end{aligned} \tag{3}$$

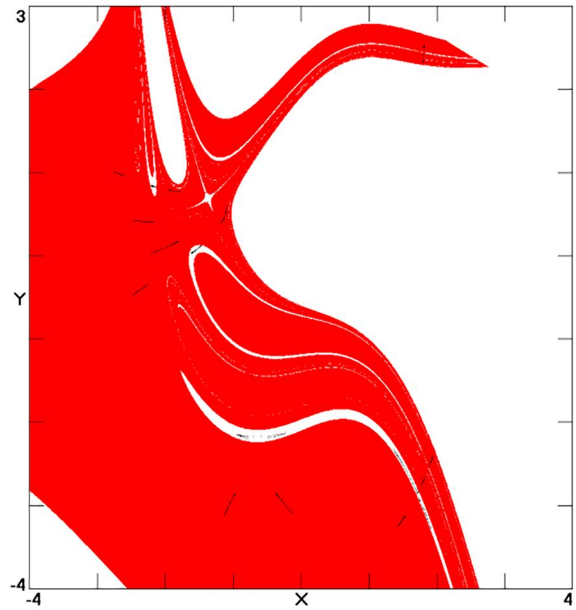

FIGURE 2 Basins of attraction of System (1) in the plane of $z = 0$

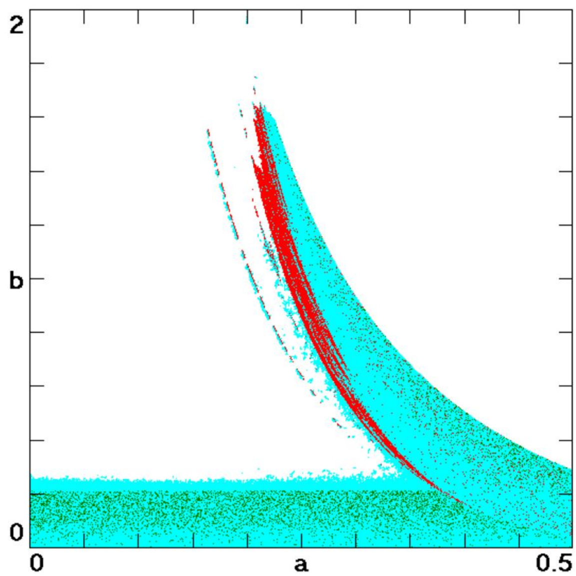

FIGURE 3 Dynamical regions in the parameter space of a, b; red (darker area if figure is rendered in black-and-white) indicates chaos, and cyan (lighter area if figure is rendereed in black-and-white) indicates limit cycle

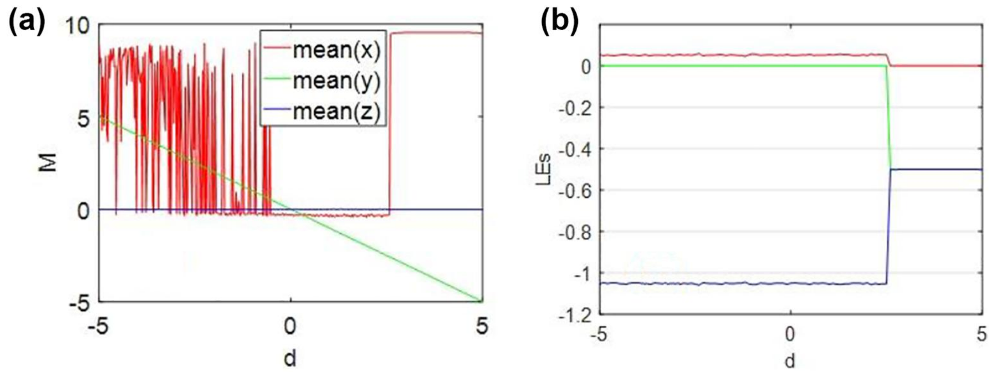

FIGURE 4 Dynamical behaviours in System (1) with $a = 0.239$, $b = 1$, and initial condition $(x_0, y_0, z_0) = (0, -2, -2)$ when the offset parameter d varies within [−5, 5] (a) Average values of variables, (b) Lyapunov exponents

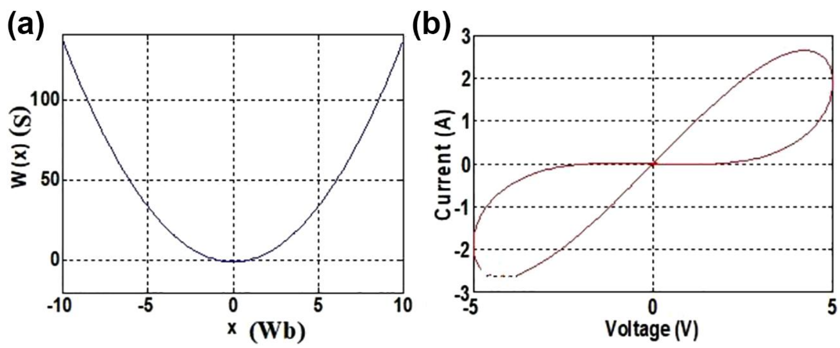

FIGURE 5 The memductance and pinched hysteresis loop

where $W_0 = 1.3((\int_{-\infty}^{t} y ds)^2 - (\int_0^t y ds)^2)$. Element memductance and the corresponding theoretical loop of pinched hysteresis are plotted in Figure 5.

3 | JERK CIRCUIT IMPLEMENTATION

Obtaining an analog circuit to realize System (1) is a relatively easy way to introduce a memristor into the operational amplifier-based integration circuit. First, we construct a 2-D jerk main structure. Second, an equivalent circuit is designed for the applied memristor without resorting to another amplifier-based integration element. The basic principle is based on the characteristics of virtual break and virtual short of an operational amplifier. An analog circuit based on Equation (1) is designed as shown in Figure 6, and according to the Kirchhoff law, the circuit equations can be written as follows:

$$\begin{cases} \dot{y} = \dfrac{1}{R_3 C_2} z, \\ \dot{z} = -\dfrac{z}{R_1 C_1} - \dfrac{z^2}{R_2 C_1} - \dfrac{W(x)y}{C_1} + \dfrac{V_b}{R_0 C_1}. \end{cases} \quad (4)$$

where the memristor W(x) is equivalent to the circuit simulator as shown in Figure 7,

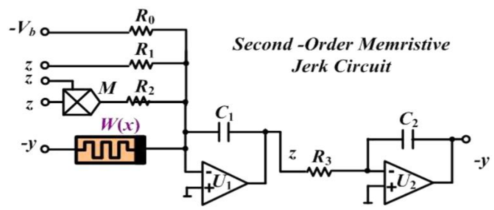

FIGURE 6 Circuit schematic of the memristive jerk oscillator

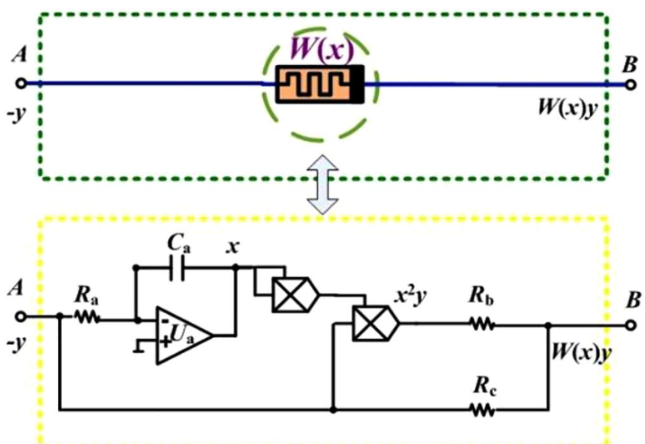

FIGURE 7 Equivalent element of the flux-dependent memristor

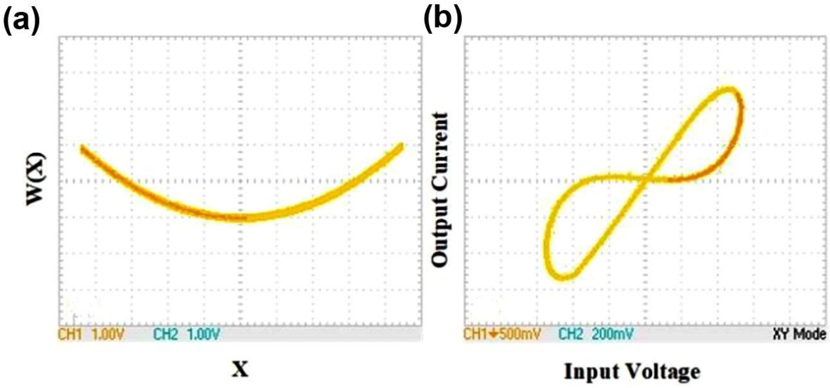

FIGURE 8 Experimental values of memductance and inherent constraints of pinched hysteresis (a) Experimental value of memductance and (b) inherent constraints of pinched hysteresis

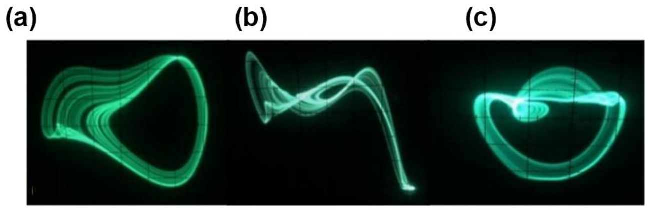

FIGURE 9 Strange attractor projections of System (4) shown in oscilloscope (a) (X, -Y) (b) (X, Z) (c) (-Y, Z) (0.5 V/div)

$$\begin{cases} i_M = W(x)y, \\ W(x) = \dfrac{1}{R_b}x^2 - \dfrac{1}{R_c}, \\ \dot{x} = \dfrac{1}{R_a C_a}y. \end{cases} \tag{5}$$

In this process, a 2-D jerk structure provides a simple structure for the proposed system. The circuit structure contains two parts based on the integration including addition and subtraction of the system variables y and z according to Equation (1). Here the analog multiplier AD633/AD is applied to realize the nonlinear product operation, while the operational amplifier OPA404/BB combined with its peripheral circuit is used to construct the addition, inversion, and integration operations. The memductance $W(x) = 1.3x^2 - 1$ is realized through the circuit parameters C_a = 100 nF, R_a = 10 kΩ, R_b = 7.69 kΩ, and R_c = 10 kΩ. The system parameters a = 0.239, b = 1 are realized through the circuit parameters R_1 = R_3 = 10 kΩ, R_2 = 41.8 kΩ, R_0 = 30 kΩ, and V_b = 3V. The capacitor C_1 = C_2 = 100 nF are selected for establishing a robust phase trajectory that only re-scales the time of the oscillation. The equations used in the circuit implementation contain different constants from those in System (1) owing to amplitude and time rescaling, which is normal in circuit implementation. Figure 8 displays a plot of the experimental constraints of memductance and inherent relation of pinched hysteresis, while Figure 9 displays the phase portraits observed in the oscilloscope.

The experimental constraints of memductance, the inherent pinched hysteresis effect, and the experimental phase portraits agree well with the numerical simulation, proving the system dynamics and the effectiveness of the hardware circuit. As a main element, the equivalent memristor can potentially have a great effect on the performance of the jerk system, which is dominantly determined by two analog multipliers and one operational amplifier. These three components define the memristor applied in this work, and therefore some other physical memristor models (e.g. the HP memristor) cannot guarantee chaos in the 3-D jerk structure.

4 | CONCLUSIONS

By introducing a memristor into a second-order jerk structure, chaotic oscillation is found in a 3-D jerk system. The proposed simple memristive jerk system has only six terms while without any equilibria, one of which is quadratic. Circuit experiments show the same oscillation, and thus they agree with the numerical simulation. When flux-controlled memductance is revised as $W(x) = 1.3\,|\,x\,|\,-1$, a minor parameter adjustment (a = 0.432, b = 1) can still recover chaos with Lyapunov exponents (0.0328, 0, −1.0332) and a corresponding attractor dimension of D_{KY} = 2.0321, which simplifies the circuit realization. Compared with other memristive systems [10,40], this system is also unique for its robust chaotic oscillation, although it is hidden [41,42]. This feature is attractive for its application in chaos-based communication or image encryption. Future work on this circuit can investigate the introduction of other memristors such as the HP memristor into this proposed jerk structure for chaos.

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